Determination of optimal sensor-actuator position for active vibration damping in collocated SISO systems using a pole-zero distance criterion for fast convergence of the search algorithm

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Abstract

The position of the transducers in active control architectures is critical to ensure the performance and has consequently been studied during the last few decades. However, the placement criteria often require the use of extensive search algorithms that demand numerous iterations, leading to prohibitive computational time for large and/or complex structures. To overcome this limitation, this paper investigates the use of the pole-zero (PZ) distance placement criterion as the starting point for a simple gradient algorithm. This open-loop criterion is based on the direct link between the PZ distance and the maximum reachable damping: the obtained position locates in the vicinity of a high damping area which ensures the convergence of the search algorithm, for fewer iterations. A numerical simulation is performed to assess the performance of the proposed approach and compared to a genetic algorithm optimization. A significant reduction of the processing time is observed while the solution shows an improved robustness to transducers misplacement.

1 Introduction

Active control strategies can be used on flexible structures that present limited damping to reduce the level of internal and/or external induced vibrations. The appropriate positioning of the transducers for those active control strategies is a well-established concern. It is indeed well-known that a poor choice of their locations can deteriorate the performance of the control architecture by inducing a lack of observability and/or controllability or, in the worth case, it can lead to stability concerns (e.g. when non-collocated sensor/actuator (SA) pairs are considered). This placement concern is highlighted by the numerous researches that have been performed during the past decades and by the availability of technical reviews that analyze the different optimal placement techniques such as [1] or [2].

The selection of the placement criterion to use among all the ones that are available in the literature is of major importance. Indeed, different SA positions can be obtained depending on the chosen objective function, influencing consequently the performance of the closed-loop system. Nevertheless, and although each criterion focuses on its specific objective (e.g. the maximization of the controllability and/or the observability, the maximization of the modal damping, the reduction of the spillover effect...), the implementation of a search algorithm is usually required to obtain the optimal solution. The aim of such an algorithm is to converge towards the best solution without computing all possible combinations, which can be prohibitive when the calculation of the objective function is expensive and/or when too many combinations are possible. The genetic algorithms (GA) are fairly suited for the optimal SA placement studies and, consequently, often considered because: (i) they are appropriate to apply in situations where non-prior knowledge is required, (ii) they can solve linear and nonlinear problems and (iii) they can find the global optimum despite the presence of multiple local ones [3]. A GA is for example applied in [4, 5] for the placement of piezoelectric actuators based on the maximization of the controllability or in [6] for the optimal sensor placement using a modified modal assurance criterion. Moreover, placement criteria in closed-loop (i.e. when the parameters of the control law are added to the optimization process) also use GAs to converge towards the optimal solution such as in [7] which seeks to reduce the average closed-loop gain or in [8, 9] that are based on linear quadratic controllers.

Despite the advantages of using a GA to obtain the optimal solution, a major drawback inherent of such a search algorithm is that it needs numerous iterations to converge, which can induce extremely long computational time for the optimization. The reason why the gradient descent method is not commonly used for the optimal placement studies is that it requires the appropriate starting parameter values to avoid obtaining a local optimum as solution. Nevertheless, if the starting values are wisely chosen (i.e. in the vicinity of the global optimal solution), the gradient descent method can be highly efficient since it requires significantly fewer evaluations to converge than the GAs. The aim of this study is to investigate the use of the PZ distance placement criterion as the starting position for a simple gradient algorithm whose purpose is to maximize the damping. The solutions are compared with the convergence of a GA in terms of performance, robustness and number of iterations. Consequently, the second part of this paper briefly recalls the basics of the gradient descent and the genetic search algorithms. The third section describes the PZ distance criterion and why such criterion is well suited for fast convergence. The fourth part provides a numerical validation on a specific case (a cantilever beam) while some conclusions are given in the last section.

2 Basics on the gradient descent and the genetic search algorithms

The GAs are stochastic algorithms inspired by the process of natural selection: starting from a randomly generated set of parameters, the ones that provide high performance with respect to the objective function are combined and mutated to produce a new generation. Then those new parameters are evaluated again and the best ones are used to produce another new generation. This procedure is stopped when one of the termination conditions is met, e.g. a defined threshold value for the average relative change in the cost function has been reached or a specified number of generations has been obtained. The different combinations (called crossover) and mutations performed by the GAs reduce the risk that the retained parameters correspond to a local optimal solution, which is one of the main advantages of these algorithms [10]. Nevertheless, because the objective function has to be evaluated for the entire population of each generation, this major advantage comes at the expense of an extremely long computational time [11].

The gradient descent method is known to be less expensive in terms of computational cost. Indeed, its convergence is based on the calculation of the gradient of the objective function and the use of this gradient to proportionally modify the value of the parameters. Once the parameters reach a local optimal solution, the gradient value tends to zero and the optimization process stops when the defined threshold is reached. Although very simple and cost-effective in terms of computational time, this method is highly sensitive to the provided starting values of the parameters to be optimized because the algorithm converges towards the local optimal solution [12]. This latest observation is the major reason why GAs are mainly used for the SA placement studies since the different criteria show a lack of prior knowledge which would allow to ensure the appropriate selection of the starting values.

Nevertheless, if the starting values are wisely chosen (i.e. in the vicinity of the global optimal solution), then the gradient descent method can be highly efficient since it requires significantly fewer evaluations to

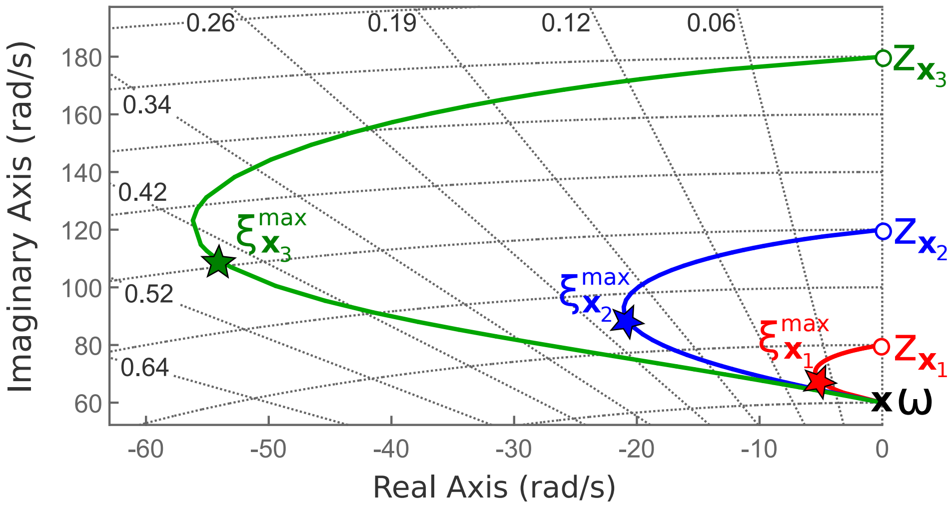


Figure 1: Graphical illustration of the link between pole-zero distance and maximum reachable damping ratio for three positions

converge than the GAs. Consequently, the use of a SA placement criterion requiring only a limited number of configurations to be tested and providing pertinent starting values for the gradient descent algorithm would be highly efficient compared to the frequently used GA convergences.

3 Description of the pole-zero distance criterion and its use for fast convergence

For active control systems, the SAs are often positioned in collocated configurations in order to benefit from the alternating poles and zeros property which ensures a 180 degrees bound of the open-loop system phase lag. In addition to this interlacing property, collocated systems exhibit a direct link between the maximum reachable damping ratio once in closed-loop and the open-loop distance between the pole and the zero (i.e. the frequency at which a non-zero actuation leads to a zero output in the response), as illustrated by Fig. 1. Consequently, such property allows to obtain preliminary closed-loop knowledge based only on open-loop information and, therefore, it can induce highly efficient estimation of the placement performance.

A new SA placement criterion based on this PZ distance concept which aims to maximize the closed-loop modal damping has first been introduced by [13] in which the cost function for the positioning of a single input single output (SISO) collocated pair is the following:

$$J(\mathbf{x}) = \left(\frac{1}{n}\sum_k \frac{|PZ(\mathbf{x})|_k}{|PZ(\mathbf{x})|_{k,max}}\right) \times \left(\sqrt[n]{\prod_k \frac{|PZ(\mathbf{x})|_k}{|PZ(\mathbf{x})|_{k,max}}}\right) \times \left(\sqrt[n]{\prod_k \frac{|P_k Z_{adj}(\mathbf{x})|}{|P_k Z_{adj}(\mathbf{x})|_{max}}}\right) \tag{1}$$

where k contains the n modes of interest, $|PZ(\mathbf{x})|_k$ is the pole-zero distance of the targeted mode at position $\mathbf{x}$ and $|P_k Z_{adj}(\mathbf{x})|$ corresponds to the adjacent pole-zero distance. The first factor aims to guarantee the overall maximization of the distance while the second factor ensures a small cost function value if one of the modes of interest presents a pole-zero cancellation (i.e. a zero distance). Finally, the third factor prevents any pole-zero cancellation with the adjacent zero of the mode of interest which could induce a loss of controllability and/or observability or a root-locus reshaping and, consequently, wrong damping predictions. The graphical representation of those three factors is shown in Fig. 2 for the second mode of the system when, due to the applied control law, the pole appears before the zero.

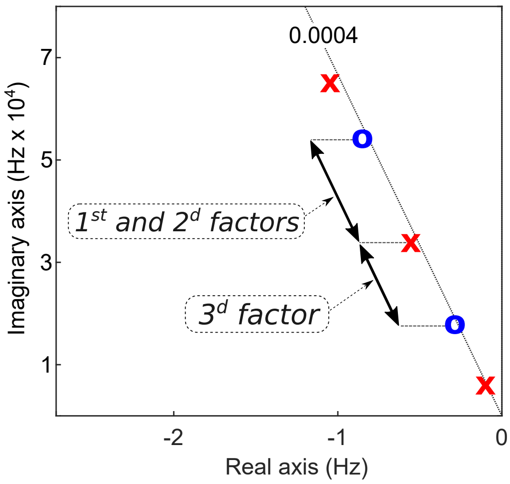


Figure 2: Graphical representation of the PZ distance criterion factors for the second mode. The interlacing property of collocated systems can be observed as well

The use of Eq. 1 as cost function for the placement of a single force-displacement collocated pair on a cantilever beam provides better damping performance compared to other open-loop criteria, as demonstrated in [13]. Nevertheless, the above-mentioned study has been performed by computing the cost function for all the possible positions, which is not acceptable when too many combinations are possible and/or when the computation of the cost function requires too much computational time. However, and unlike the other placement criteria, the PZ distance criterion is based on a direct and physical link between open-loop knowledge and closed-loop behavior (respectively the PZ distance and the maximum reachable damping). Moreover, all the zeros are always bounded between two surrounding eigen frequencies due to the interlacing property of collocated systems and their values continuously evolve within the bounded intervals. Because of those properties, it is clear that only a limited number of positions $\mathbf{x}$ would need to be computed by Eq. 1 to produce a preliminary trend of the PZ distance. The position among the limited tested ones that maximizes the distance could therefore be defined as the starting value for the gradient convergence algorithm, allowing consequently its usage which is not feasible with other criteria due to the lack of prior knowledge.

Furthermore and considering the aforementioned properties of the zeros, the number of limited positions to be computed by the PZ criterion can be directly proportional to the wavelength of the targeted mode shape(s) of interest. Nonetheless, a straightforward wavelength value of a mode shape is not always easily defined, which is for example the case when considering a cantilever beam. Therefore, and to remain as general as possible, the number of limited positions to be computed by the PZ criterion can be directly proportional to the mode ID. This is consistent with the dynamics of a vibrating system: the higher the mode ID, the smaller its mode shape wavelength and, consequently, the higher the number of points required to accurately describe the dynamics. Hence, if the mode ID is defined as i, the number of limited positions to be computed by the PZ criterion r can be obtained by Eq. 2 where γ is the proportionality factor.

$$r_i = \gamma \times i \tag{2}$$

Accordingly, the PZ distance criterion provided by Eq. 1 requires only r_i times its evaluations to obtain a first trend of the PZ distance of mode i as a function of the SA pair positions. Due to the physical link between the PZ distance and the maximum reachable damping, the position that provides the maximum obtained distance can be defined as the starting value for the gradient search algorithm, which will converge towards the optimum that locates in its vicinity. Therefore, and unlike the GA convergence, only a small amount of iterations is needed, reducing considerably the overall processing time for the optimization study. In order to validate this approach, it is applied on a cantilever beam in the next section in which the optimal placement of a collocated force actuator/displacement sensor pair is studied.

4 Application to a cantilever beam

The proposed strategy is numerically illustrated on a cantilever beam for the optimal placement of a collocated force actuator/displacement sensor pair whose purpose is to separately maximize the modal damping of the first three bending modes. The beam is modeled using the Structural Dynamics Toolbox (an open and extendable finite element modeling Matlab based toolbox for dynamics problems [14]) with the following properties: dimensions of 300 mm x 25 mm x 2 mm and use of a lightly damped steel (Young's modulus E = 210 GPa, Poisson's ratio ν = 0.285, mass density ρ = 7800 kg/m^3 and modal damping ratio ξ = 0.004). Because only the bending modes of the beam are targeted, the finite element model is built using beam elements that allow only the 2 degrees of freedom out-of-plane bending motion. The mesh accuracy in the longitudinal direction is set to 0.1 mm which leads to a total of 3001 elements, each one being a potential SA pair position except for the constrained node at the clamped end of the cantilever beam. The convenience to apply such high accuracy for the mesh size is that it provides an extensive number of positioning candidates while using a simple demonstration structure.

As described in the previous section, the proposed approach consists of (1) computing the PZ distance criterion of Eq. 1 for r positions and (2) using the position that maximizes the computed distances as starting value for the gradient descent algorithm. The number of position r is obtained by Eq. 2 in which the proportionality factor γ is defined to 10 for this study. It can be noted that a more advanced selection of this value could be accomplished by, for example, performing a sensitivity analysis of the zeros with respect to the SA position. Because the current validation focuses on the benefits of applying the proposed approach only, this additional sensitivity analysis could be performed in a separate future work.

The aim of the optimization being to separately maximize the modal damping of the first bending modes, the search algorithm can directly focus on the closed-loop performance by applying the cost function $\tilde{J}$ defined by Eq. 3, where ξ_i is the closed-loop modal damping of mode i and g is the gain value of the applied controller. Consequently, the gradient descent algorithm will perform the optimization on two parameters: the SA position $\mathbf{x}$ and the controller gain g.

$$\tilde{J}(\mathbf{x}, g) = \xi_i \tag{3}$$

Regarding the controller, a lead control law as described by Eq. 4 is considered where the zero z_c and the pole p_c only depend on the targeted mode as follow: $z_c = \omega_i/2$ and $p_c = \omega_i \times 2$ where ω_i is the natural pulsation of the mode i. Because the PZ distance criterion does not deal with the optimal gain, the starting value of g is set to $g_{start} = z_i \times \sqrt{z_i/\omega_i}$ where z_i is the zero value of the ith mode at the considered position. This starting gain value is directly based on the analytic gain that provides the maximum reachable damping for a lead control as described in [15].

$$C(s) = g\,\frac{s + z_c}{s + p_c} \tag{4}$$

Since the controller has been characterized, it is possible to properly define the PZ distance: because the pole appears before the zero when applying a lead controller, the PZ distance is computed by subtracting the pole to the zero [15]. Applying the PZ distance criterion of Eq. 1 at r_i locations obtained by steps of equal size for the first three bending modes provides the following positions from the clamped end of the beam: $0.778 \times L$, $0.474 \times L$ and $0.345 \times L$ for the first, second and third modes respectively where L is the length of the beam. Each of those results is used as the starting position for the gradient descent algorithm whose cost function is defined by Eq. 3 for the separate control of the first three modes.

The performance obtained with the proposed approach can be compared to the ones obtained with the commonly used genetic search algorithm. Consequently, the GA available within Matlab is implemented with the following properties: the cost function to maximize is given by Eq. 3 using the same control law described

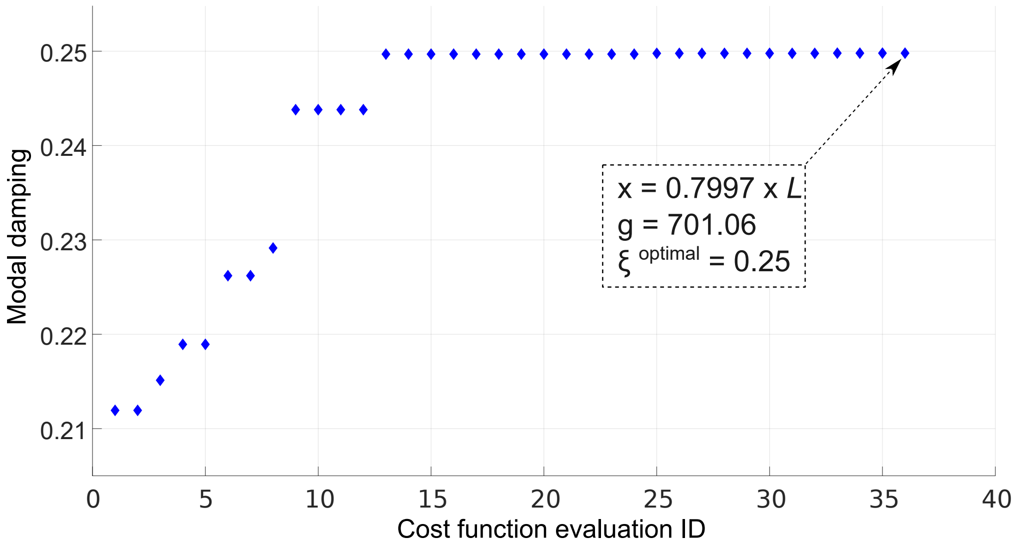


Figure 3: Convergence of the gradient descent algorithms for the first bending mode of the beam

by Eq. 4, the population size per generation is set to 50, the crossover fraction is defined to 0.8 and a random mutation from bounded Gaussian distributions is used.

The comparison between the two approaches is available in Tables 1 and 2 where the optimal positions and the related optimal gain values for each mode are available, for the two tested algorithms. Moreover, the corresponding closed-loop modal damping and the total number of evaluations of the cost function performed by the two algorithms are available as well. Because the proposed approach implies r_i computations of the cost function $J(\mathbf{x})$ to determine the starting position as well as the evaluation of $\tilde{J}(\mathbf{x}, g)$ for the convergence of the gradient descent, the total number provided in Table 1 combined those two evaluation numbers. Nevertheless and as it can be observed, the GA convergence demands significantly more evaluations than the gradient descent to achieve convergence. There is indeed a factor 7 for the optimization of the first mode and this factor goes up to 10 for the third mode convergence.

This faster convergence is highlighted by the comparison between Figs. 3 and 4 that respectively show the modal damping convergence obtained with the gradient descent optimization and the GA for the first bending mode. As it can be seen, the gradient search algorithm presents several stationary levels with respect to the damping value. Those phases correspond to the few evaluations the algorithm requires to determine the convergence directions. Because the different establishments of the directions are straightforward, the optimization needs only 36 evaluations of $\tilde{J}(\mathbf{x}, g)$ in total to converge. Conversely, the GA is based on a stochastic approach: a population of 50 members is first randomly initiated. This initial population is then entirely evaluated in order to select the adequate members to form the new evolved generation by mutations and crossovers. The convergence consequently emerges only after the evaluation of the entire population when the new generation is created, as it can be seen in Fig. 4 where each generation of 50 members is delimited by an alternating background color. As a result, the algorithm requires numerous evaluations of the cost function which leads to a total of 321 to converge.

Table 1: Optimal results obtained with the gradient descent convergence.

	Gradient descent optimization			
	Normalised optimal position (/L)	Optimal gain	Optimal damping	Evaluations
Mode 1	0.7997	701.06	0.25	46
Mode 2	0.5003	21219	0.23	65
Mode 3	0.3553	103101	0.22	84

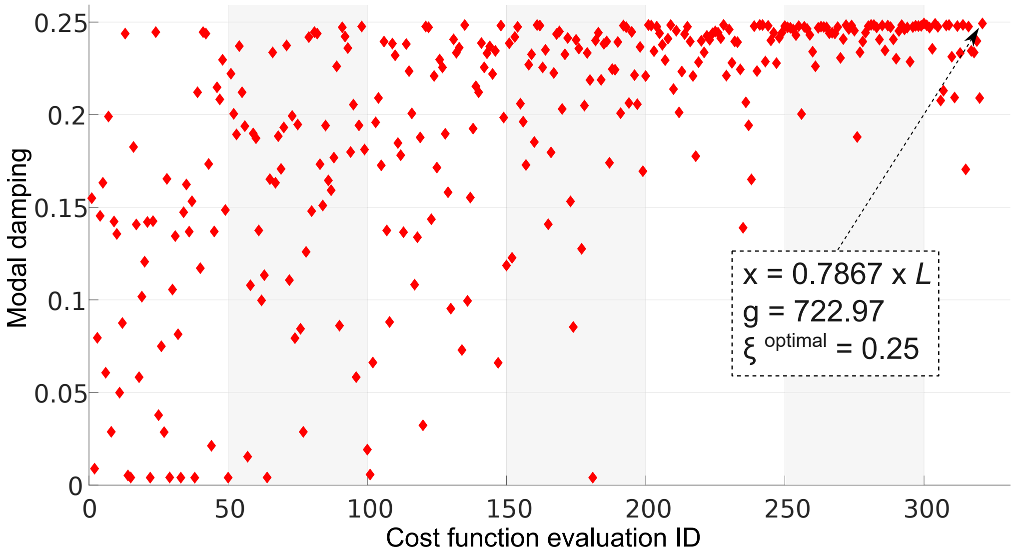


Figure 4: Convergence of the genetic algorithm for the first bending mode of the beam

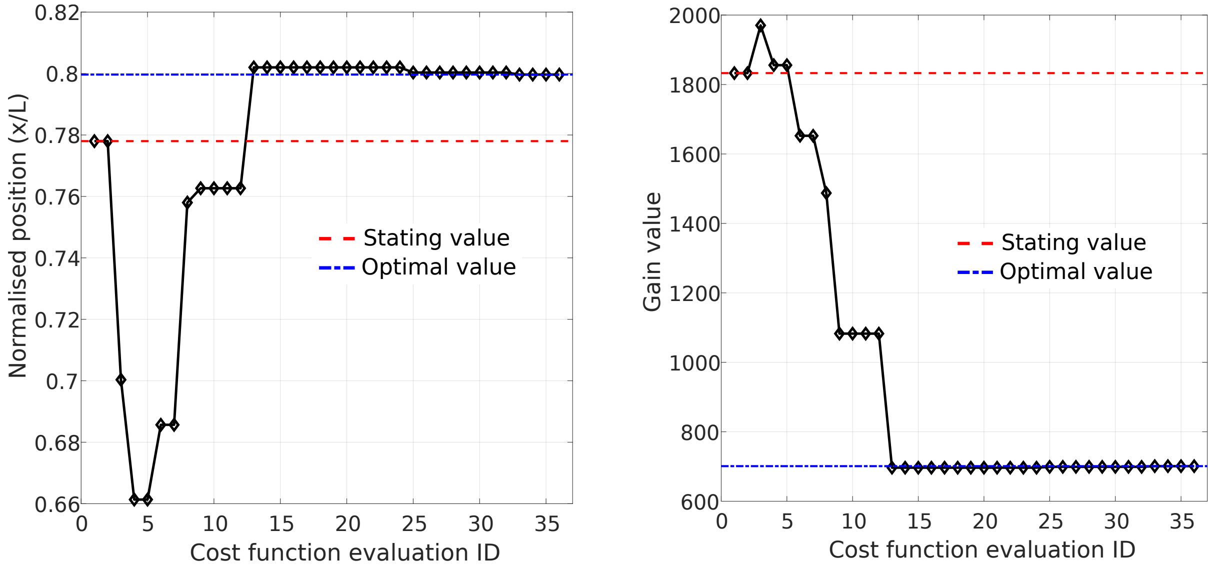
Table 2: Optimal results obtained with the genetic algorithm convergence

	Genetic algorithm optimization			
	Normalised optimal position (/L)	Optimal gain	Optimal damping	Evaluations
Mode 1	0.7867	722.97	0.25	321
Mode 2	0.8570	16135	0.35	561
Mode 3	0.9287	56306	0.34	881

Interestingly, it can be observed that the gradient descent provides optimal positions close to the starting ones obtained with the PZ distance criterion (i.e. $0.778 \times L$, $0.474 \times L$ and $0.345 \times L$ for the first, second and third modes respectively). The proximity between the starting positions and the optimal ones is illustrated by Fig. 5 which shows the convergence of the two parameters during the gradient descent optimization of the first mode. This highlights therefore the efficiency of the PZ criterion that only requires a small number of positions to be tested while ensuring the selection of a solution in the vicinity of high damping area. Moreover, it also indicates that the gradient convergence is mainly employed to determine the optimal gain value. Consequently, the selection of the starting gain value could be improved which would decrease the total number of evaluations and reduce even further the computational time. Such an improvement could be considered in a future work. The convergence of the SA pair position and the gain value during the GA optimization of mode 1 is given in Fig. 6 for comparison purpose.

Although the proposed method allows faster convergence, it can be seen that the obtained solutions for modes 2 and 3 provide lower damping than the GA optimization. This therefore means that the PZ distance criterion induces the convergence towards local optima for the second and third modes while the GA correctly converges towards the global solutions. Nevertheless, even though the maximization of the damping is the main objective of the optimization, special attention should be paid to the robustness of the obtained solutions. More particularly, the robustness to transducer misplacement is of high importance because such misplacement is indeed common to observe when the SAs are mounted on real-life structures. Consequently, a sensitivity analysis of the damping with respect to the transducer misplacement can be performed to assess the robustness of the optimal positions.

The sensitivity analysis is performed as follows: for the three solutions obtained for each mode and with the two algorithms, the SA pair is moved along the longitudinal axis of the beam around the optimal position. A misplacement up to 1% of the length of the beam is chosen in both directions, leading to a positioning error



(a) SA pair position (b) Gain value

Figure 5: Convergence of the gradient descent algorithm for the first bending mode for: (a) the SA pair position, (b) the gain value

interval of ±3mm from the optimal positions. Naturally, the optimal gains as well as the control law are kept without any changes. To ease the visualization and the comparison of the different sensitivities (i.e. between the three modes and the two algorithms), the relative positioning error $\Delta\mathrm{x}$ and the relative modal damping error $\Delta\xi$ are introduced as follows:

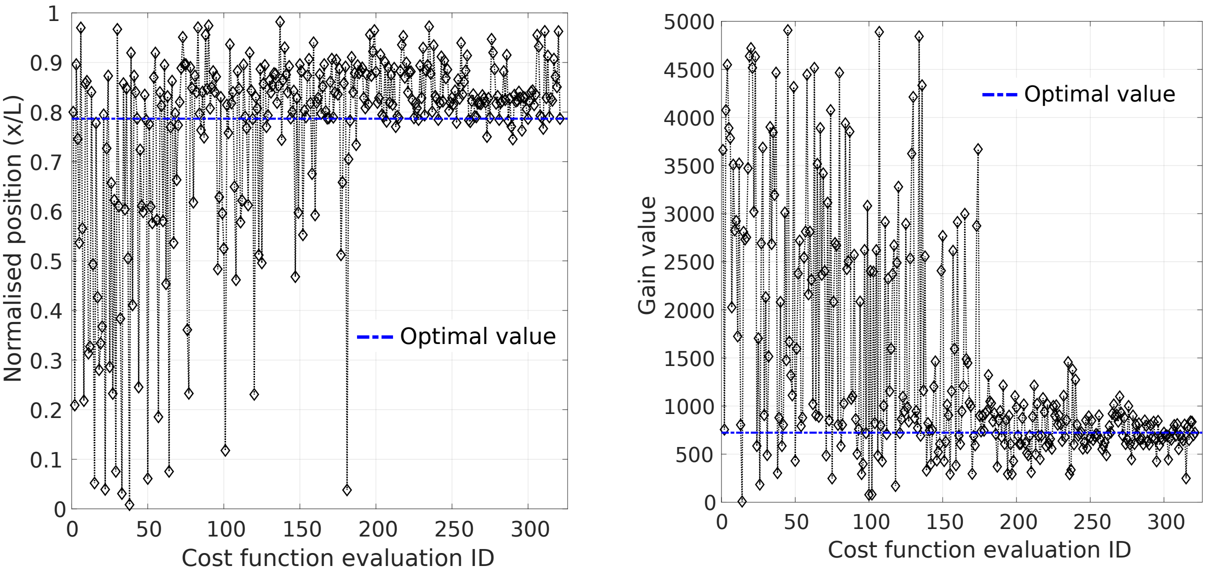
$$\Delta\mathrm{x} = \frac{x - x_{optimal}}{L} \tag{5}$$

$$\Delta\xi = \frac{\xi - \xi_{optimal}}{\xi_{optimal}} \tag{6}$$

where $\Delta\mathrm{x}$ is defined as the ratio between the positioning error and the length of the beam and $\Delta\xi$ corresponds to the ratio between the modal deviation and the optimal damping value.

The results obtained with the sensitivity analysis are shown in Figs. 7a and 7b, respectively for the gradient descent and for the GA convergence. As it can be seen, the optimal positions obtained with the PZ distance criterion and the gradient descent algorithm show a strong resilience to misplacement. Indeed, the relative damping error $\Delta\xi$ does not exceed 2% over the full misplacement range. On the contrary, the optimal positions obtained with the GA convergence present a damping error deviation down to 80%, as illustrated by Fig. 7b. Such deviation can be explained by the fact that the optimal positions found by the GA for the second and third modes locate in the vicinity of a complete root-locus reshaping due to (1) the pole/zero cancellation of a surrounding mode and (2) the position of a zero near the pole of the targeted mode. Hence, as soon as the positioning error reaches such area, the dynamics changes and the damping abruptly drops.

This situation is for example illustrated by Fig. 8 where the maximum reachable modal damping of mode 2 is shown. Such curve has been obtained by extensively computing, for each possible SA position along the longitudinal axis, the maximum reachable damping obtained with the optimal gain. The starting point provided to the gradient descent as well as the optimal results obtained with the two algorithms are plotted as well. As it can be observed, the GA precisely reaches the global optimum while, due to the starting point obtained with the PZ criterion, the gradient descent algorithm converges to a local optimum. Nevertheless and as stated, the global optimum obtained with the GA locates right next to a major drop of the damping, confirming therefore the poor misplacement robustness described by Fig. 7b. Interestingly, the positions that



(a) SA pair position

(b) Gain value

Figure 6: Convergence of the GA for the first bending mode for: (a) the SA pair position, (b) the gain value

stand in the vicinity of such a situation are directly rejected by the PZ distance criterion thanks to its third term which analyzes the dynamics of the adjacent mode, as illustrated by Fig. 2.

Consequently, and although the damping values obtained with the proposed approach are slightly below the GA ones, they present high robustness to misplacement while ensuring good damping performance because of the valuable information contained within the criterion.

5 Conclusions

This paper presents a new approach for the determination of the optimal SA position in collocated SISO systems. This approach is based on the computation of the PZ distance criterion on a limited number of locations in order to obtain a starting position for the gradient descent optimization algorithm. The PZ distance criterion is based on the direct link between open-loop PZ distance and closed-loop damping, which ensures that its solution stands in the vicinity of a high-damping location. This consequently ensures the proper convergence of the search algorithm while requiring limited iterations. It is numerically proven with the optimal placement study of a collocated force actuator/displacement sensor pair that the proposed approach decreases the required number of evaluations up to a factor 10 with respect to the commonly used GA convergence. Moreover, and despite the slightly lower damping values obtained with the proposed approach, it is shown that the optimal solutions present high robustness with respect to transducer misplacement unlike the solutions obtained with the GA.

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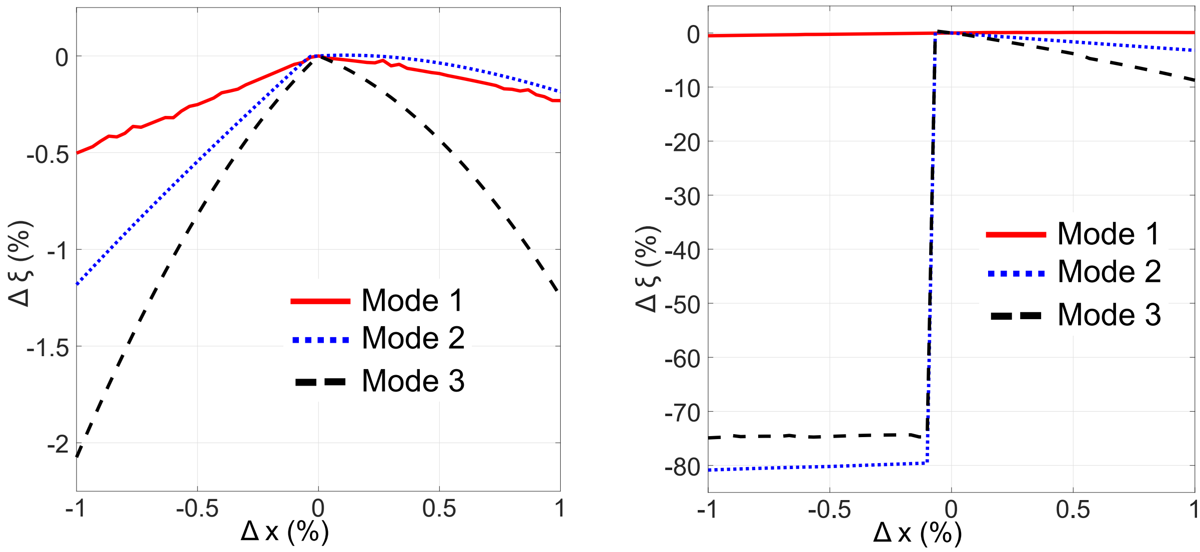

(a) Gradient algorithm

(b) Genetic algorithm

Figure 7: Sensitivity analysis to the SA pair misplacement when considering the optimal positions obtained with: (a) the gradient descent algorithm, (b) the genetic algorithm

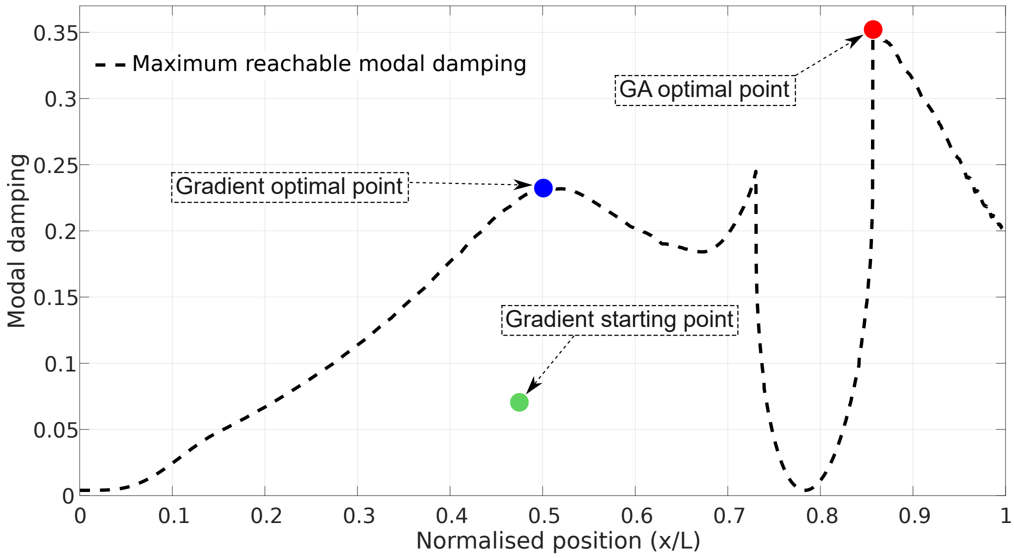

Figure 8: Projection of the maximum reachable modal damping for mode 2 on the longitudinal axis of the beam

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